

Simultaneous expansion and rotation of shear-free universes in modified gravity

Amare Abebe*, Rituparno Goswami* and Peter K.S. Dunsby*,[†]

* *Astrophysics, Cosmology and Gravity Centre (ACGC), University of Cape Town, Rondebosch, 7701, South Africa*

[†] *South African Astronomical Observatory, Observatory, 7925, South Africa*

Abstract. We show in a fully covariant way that, there exist a class of $f(R)$ models for which a shear-free, almost FLRW universe can expand and rotate at the same time .

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INTRODUCTION

The action of a generalized fourth-order gravity is given by (for more details, see [1] and references therein):

$$\mathcal{A} = \frac{1}{2} \int d^4x \sqrt{-g} [f(R) + 2\mathcal{L}_m] , \quad (1)$$

where \mathcal{L}_m represents the matter contribution, and the generalized field equations read

$$G_{ab} = \tilde{T}_{ab}^m + T_{ab}^R \equiv T_{ab} , \quad (2)$$

where

$$\tilde{T}_{ab}^m = \frac{T_{ab}^m}{f'} , \quad T_{ab}^R = \frac{1}{f'} \left[\frac{1}{2}(f - Rf')g_{ab} + \nabla_b \nabla_a f' - g_{ab} \nabla_c \nabla^c f' \right] . \quad (3)$$

Here we have defined $f' \equiv df(R)/dR$ and $T_{ab}^m = \mu^m u_a u_b + p^m h_{ab} + q_a^m u_b + q_b^m u_a + \pi_{ab}^m$, where μ^m , p^m , q^m and π_{ab}^m denote the standard matter density, pressure, heat flux and anisotropic stress respectively.

The *total* thermodynamics of the matter-curvature composite is then given by

$$\mu \equiv \frac{\mu^m}{f'} + \mu^R , \quad p \equiv \frac{p^m}{f'} + p^R , \quad q_a \equiv \frac{q_a^m}{f'} + q_a^R , \quad \pi_{ab} \equiv \frac{\pi_{ab}^m}{f'} + \pi_{ab}^R , \quad (4)$$

where μ^R , etc. are thermodynamical quantities of the curvature fluid defined in the next section. The covariant derivative of a timelike vector u^a can be decomposed into basic parts as

$$\nabla_a u_b = -A_a u_b + \frac{1}{3} h_{ab} \Theta + \sigma_{ab} + \varepsilon_{abc} \omega^c , \quad (5)$$

where $A_a = \dot{u}_a$ is the acceleration, $\Theta = \tilde{\nabla}_a u^a$ is the expansion, $\sigma_{ab} = \tilde{\nabla}_{\langle a} u_{b \rangle}$ is the shear tensor and $\omega^a = \varepsilon^{abc} \tilde{\nabla}_b u_c$ is the vorticity vector. For the Weyl curvature tensor one has

$$E_{ab} = C_{abcd} u^c u^d = E_{\langle ab \rangle} , \quad H_{ab} = \frac{1}{2} \varepsilon_{acd} C_{be}^{cd} u^e = H_{\langle ab \rangle} , \quad (6)$$

giving a covariant description of *tidal forces* and *gravitational radiation* respectively.

LINEARIZED FIELD EQUATIONS

We consider the background to be Friedmann-Lemăître-Robertson-Walker (FLRW), where the Hubble scale sets the characteristic scale of the perturbations. In the perturbed spacetime the standard matter is considered to be a perfect fluid with the energy momentum tensor given by:

$$T_{ab}^m = (\mu^m + p^m)u_a u_b + p^m g_{ab} . \quad (7)$$

with $p^m = w\mu^m$ and the heat flux (q_a^m) and the anisotropic stress (π_{ab}^m) vanishing in the perturbed spacetime. In addition, since we consider shear-free perturbations, the shear tensor σ_{ab} vanishes identically.

For the *curvature fluid* the linearized thermodynamic quantities are given by

$$\begin{aligned} \mu^R &= \frac{1}{f'} \left[\frac{1}{2}(Rf' - f) - \Theta f'' \dot{R} + f'' \tilde{\nabla}^2 R \right] , \\ p^R &= \frac{1}{f'} \left[\frac{1}{2}(f - Rf') + f'' \dot{R} + f''' \dot{R}^2 + \frac{2}{3} (\Theta f'' \dot{R} - f'' \tilde{\nabla}^2 R) \right] , \\ q_a^R &= -\frac{1}{f'} \left[f''' \dot{R} \tilde{\nabla}_a R + f'' \tilde{\nabla}_a \dot{R} - \frac{1}{3} f'' \Theta \tilde{\nabla}_a R \right] , \quad \pi_{ab}^R = \frac{1}{f'} f'' \tilde{\nabla}_{\langle a} \tilde{\nabla}_{b \rangle} R . \end{aligned} \quad (8)$$

With the conditions above, the propagation and constraint equations can be given by

$$\dot{\Theta} - \tilde{\nabla}_a A^a = -\frac{1}{3}\Theta^2 - \frac{1}{2}(\mu + 3p) , \quad (9)$$

$$(\omega^{\langle a \rangle})_{;c} - \frac{1}{2}\varepsilon^{abc} \tilde{\nabla}_b A_c = -\frac{2}{3}\Theta \omega^a , \quad (10)$$

$$E^{\langle ab \rangle} - \varepsilon^{cd\langle a} \tilde{\nabla}_c H_d^{\rangle b} = -\Theta E^{ab} - \frac{1}{2}\pi_R^{ab} - \frac{1}{2}\tilde{\nabla}^{\langle a} q_R^{b \rangle} - \frac{1}{6}\Theta \pi_R^{ab} , \quad (11)$$

$$H^{\langle ab \rangle} + \varepsilon^{cd\langle a} \tilde{\nabla}_c E_d^{\rangle b} = -\Theta H^{ab} + \frac{1}{2}\varepsilon^{cd\langle a} \tilde{\nabla}_c \pi_{dR}^{\rangle b} , \quad (12)$$

$$\dot{\mu}_m = -(\mu_m + p_m)\Theta , \quad (13)$$

$$\dot{\mu} + \tilde{\nabla}^a q_a^R = -(\mu + p)\Theta ; \quad (14)$$

$$(C_0)^{ab} := E^{ab} - \tilde{\nabla}^{\langle a} A^{b \rangle} - \frac{1}{2}\pi_R^{ab} = 0 , \quad (15)$$

$$(C_1)^a := \tilde{\nabla}^a \Theta - \frac{3}{2}\varepsilon^{abc} \tilde{\nabla}_b \omega_c - \frac{3}{2}q_R^a = 0 , \quad (16)$$

$$(C_2) := \tilde{\nabla}^a \omega_a = 0 , \quad (17)$$

$$(C_3)^{ab} := H^{ab} + \tilde{\nabla}^{\langle a} \omega^{b \rangle} = 0 , \quad (18)$$

$$(C_4)^a := \tilde{\nabla}^a p_m + (\mu_m + p_m)A^a = 0 , \quad (19)$$

$$(C_5)^a := \tilde{\nabla}_b E^{ab} + \frac{1}{2}\tilde{\nabla}_b \pi_R^{ab} - \frac{1}{3}\tilde{\nabla}^a \mu + \frac{1}{3}\Theta q_R^a = 0 , \quad (20)$$

$$(C_6)^a := \tilde{\nabla}_b H^{ab} + (\mu + p)\omega^a + \frac{1}{2}\epsilon^{abc}\tilde{\nabla}_b q_c^R = 0. \quad (21)$$

The conditions $\sigma_{ab} = 0$ and $q_m^a = 0$ give the two new constraints $(C_0)^{ab}$ and $(C_4)^a$ respectively. Substituting $(C_0)_{bd}$ into $(C_5)_b$ and using $(C_4)_b$ we obtain the constraint

$$\frac{w}{w+1}\tilde{\nabla}^d\tilde{\nabla}_{\langle b}\tilde{\nabla}_{d\rangle}\phi + \frac{1}{3}\tilde{\nabla}_b\mu - \tilde{\nabla}^d\pi_{bd}^R - \frac{1}{3}\Theta q_b^R = 0, \quad (22)$$

where $\phi \equiv \ln \mu_m$. To check the spatial consistency of the above constraint on any initial hypersurface we take the curl of (22) to obtain

$$\omega^a \left[\left(\frac{w\Theta}{3} + \frac{\dot{R}f''}{3f'} \right) \tilde{R} + \frac{2(1+w)\mu_m\Theta}{3f'} \right] + \left(\frac{\dot{R}f''}{f'} + w\Theta \right) \tilde{\nabla}^2 \omega^a = 0, \quad (23)$$

where $\tilde{R} = 2(\mu - \frac{1}{3}\Theta^2)$. Now defining the expansion, acceleration, jerk and snap parameters by the following relations

$$\Theta = 3\frac{\dot{a}}{a}, \quad q = -\frac{\ddot{a}a}{\dot{a}^2}, \quad j = \frac{a^2}{\dot{a}^3}\frac{d^3a}{dt^3}, \quad s = \frac{a^3}{\dot{a}^4}\frac{d^4a}{dt^4}, \quad (24)$$

and using

$$\begin{aligned} \dot{\Theta} &= -\frac{1}{3}\Theta^2(1+q), \quad \dot{q} = -\frac{1}{3}\Theta(j-q-2q^2), \quad \ddot{\Theta} = \frac{1}{9}\Theta^3(2+3q+j), \\ \dot{j} &= \frac{1}{3}\Theta(s+2j+3qj), \quad \ddot{q} = -\frac{1}{9}\Theta^2[s+2j-3q^2+6qj-6q^3], \quad \dot{R} = \frac{2}{3}\Theta Q, \end{aligned} \quad (25)$$

where

$$Q = \frac{1}{3}\Theta^2(j-q-2)+\tilde{R}, \quad \dot{Q} = \frac{1}{9}\Theta[(4+5q+j+jq+s)\Theta^2+6\tilde{R}], \quad (26)$$

we can rewrite (23) as

$$\frac{2}{3}\Theta \left\{ \omega^a \left[\left(\frac{w}{2} + \frac{f''}{3f'}Q \right) \tilde{R} + \frac{(1+w)\mu_m}{f'} \right] + \left[\frac{f''}{f'}Q + \frac{3w}{2} \right] \tilde{\nabla}^2 \omega^a \right\} = 0. \quad (27)$$

Spatial consistency requires the vanishing of either Θ or the terms in the curly brackets. For temporal consistency differentiate (27) w.r.t. time to get

$$\Theta \omega^a \left\{ \left[\frac{(1-w)P}{3}\tilde{R} + \frac{(1+w)}{f'} \frac{(3w+5)f' + 4f''Q}{6f'} \mu_m \right] + \frac{Z}{P} \left[\left(\frac{1+w}{f'} \right) \mu_m \right] \right\} = 0, \quad (28)$$

where

$$Z = \frac{2}{3} \left(\frac{f'''}{f'} - \left(\frac{f''}{f'} \right)^2 \right) Q^2 + \frac{f''}{9f'} ((4+5q+j+jq+s)\Theta^2 + 6\tilde{R}). \quad (29)$$

It follows that for the new constraints to be spatially and temporally consistent we must have either $\Theta\omega^a = 0$ or the expression in the curly brackets must vanish. It is easy to see

from (28) that if the 3-curvature vanishes, then $\Theta\omega^a = 0$ for vacuum universes ($\mu_m = 0$). This implies that *a shear-free, spatially flat vacuum universe in any $f(R)$ theory can rotate and expand simultaneously in the linearized regime.*

In the non - vacuum case, there exists at least one non-trivial case which does violate the Ellis condition. For a flat Milne universe, the Friedmann equation is given by

$$-R^2 \frac{d^2 f(R)}{dR^2} + \frac{f(R)}{2} - \frac{\mu_0}{a(R)^{3(1+w)}} = 0, \quad (30)$$

and has the following general solution:

$$f(R) = C_1 R^{\frac{1+\sqrt{3}}{2}} + C_2 R^{\frac{1-\sqrt{3}}{2}} - \frac{4\mu_0}{1+12w+9w^2} R^{\frac{3(1+w)}{2}}. \quad (31)$$

Considering the particular solution ($C_1 = 0 = C_2$), and comparing it with Eqn. (28), for the corresponding flat Milne universe in R^n gravity, we obtain

$$\frac{(1+w)\mu_m}{6f'} [3w+9-4n] = 0. \quad (32)$$

Comparing solutions (32) and the particular solution of (31) (with $n = 3(1+w)/2$) we find that $w = 1$ if $\mu_m \neq 0$. In other words, *for a stiff fluid in R^3 gravity, there exists a flat Milne-universe solution which can rotate and expand simultaneously at the level of linearized perturbation theory.*

DISCUSSION AND CONCLUSION

In this work we showed that if the 3-curvature vanishes, then the result of [2] can always be avoided for vacuum universes. We also demonstrated there is at least one physically realistic non-vacuum case in which both rotation and expansion are simultaneously possible. This suggests that there are situations where linearized fourth-order gravity shares properties with Newtonian theory not valid in General Relativity.

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